Northern Regional Hub-funded project

e-Book

Selected puzzles, paradoxes and sophisms for tertiary STEM students

Sergiy Klymchuk, Sergei Gulyaev

2017

Contents

AOTEAROA
NATIONAL CENTRE FOR
TERTIARY TEACHING
EXCELLENCE

An Ako Aotearoa publication.
This project output has been funded by
Ako Aotearoa through the
Regional Hub Project Fund.

Chapter 1

Paradoxes and Sophisms in Engineering Mathematics

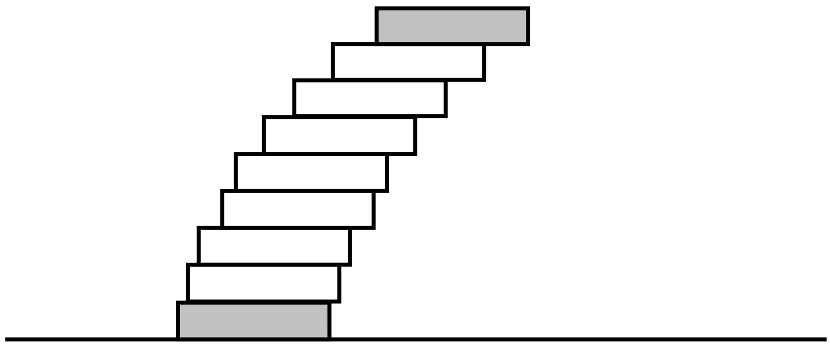
This chapter of the e-book presents paradoxes and sophisms for first year engineering mathematics courses. The word *paradox* comes from the Greek word *paradoxon* which means *unexpected*. There are several usages of this word, including those that deal with contradiction. In this chapter, the word means a surprising, unexpected, counter-intuitive statement that *looks* invalid but in fact is *true*. The word *sophism* comes from the Greek word *sophos* which means *wisdom*. In modern usage it denotes intentionally invalid reasoning that *looks* formally correct, but in fact contains a subtle mistake or flaw. In other words, it is a *false* proof of an incorrect statement. Each such 'proof' contains some sort of error in reasoning. Often finding and analyzing the mistake in a sophism can give a student deeper understanding than a recipe-based approach in solving standard problems. Most students are exposed to sophisms at school. Some basic examples of the sophisms at the school level use division by zero or taking only a non-negative square root to 'prove' statements like "1 = 2". In this book, to 'prove' such statements most of the tricks use *Calculus*.

Much of the chapter's content can be used as edutainment – both education and entertainment. The intention is to engage students' emotions, creativity and curiosity and also enhance their conceptual understanding, critical thinking skills, problem-solving strategies and lateral thinking "outside the box". Many would agree if students are interested the rest is easy.

1.1 Paradoxes

1. Laying bricks

   Imagine you have an unlimited amount of the same ideal homogeneous bricks. You are constructing an arc by putting the bricks one on top of another without using any cementing solution between them. Each successive brick is further to the right than the previous (see the diagram below).



How far past the bottom brick can the top brick extend?

2. Spiral curves

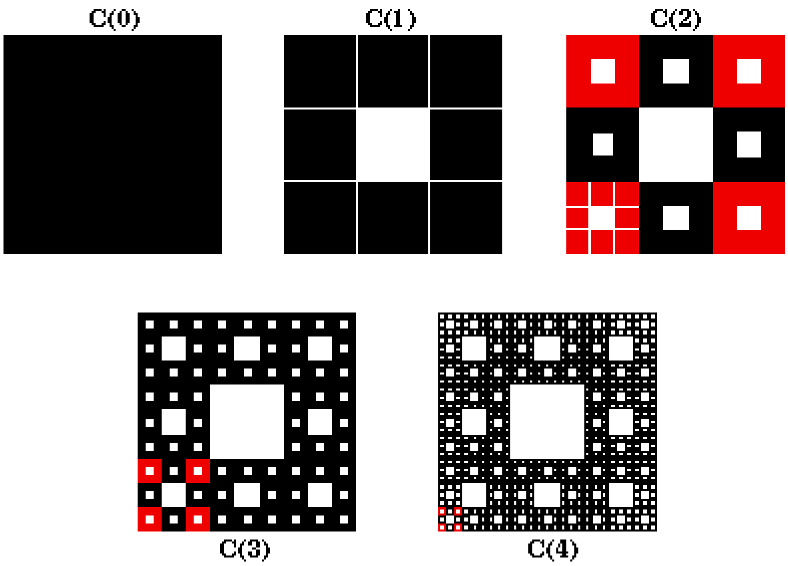
   Construct two similar-looking spiral curves that both rotate infinitely many times around a point, with one curve being of a finite length and the other of an infinite length.

3. A tricky curve

   Construct a curve that is closed, not self-crossing, has an infinite length and is located between two other closed, not self-crossing curves of a finite length.

4. A tricky area

   A square with sides of 1 unit (and therefore area of 1 square unit) is divided into 9 equal squares, each with sides 1/3 unit and areas of 1/9 square unit, then the central square is removed. Each of the remaining 8 squares is divided into 9 equal squares and the central squares are then removed. The process is continued infinitely many times. The diagram below shows the first 4 steps. At every step 1/9 of the current area is removed and 8/9 is left, that is at every step the remaining area is 8 times bigger than the area removed. After infinitely many steps what would the remaining area be?

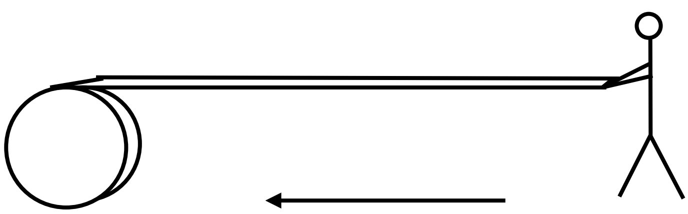


5. A tricky shape

   It looked like the cross-section of an object of circular shape. To determine whether it was a circle, a student suggested measuring its several diameters (the length of line segments passing through the centre of symmetry of the figure and connecting its two opposite boundary points). The student reasoned that if they all appeared the same, the object had a circular shape. Was the student's reasoning correct?

6. Rolling a barrel

   A person holds one end of a wooden board 3 m long and the other end lies on a cylindrical barrel. The person walks towards the barrel, which is rolled by the board sitting on it. The barrel rolls without sliding. This is shown in the following diagram:

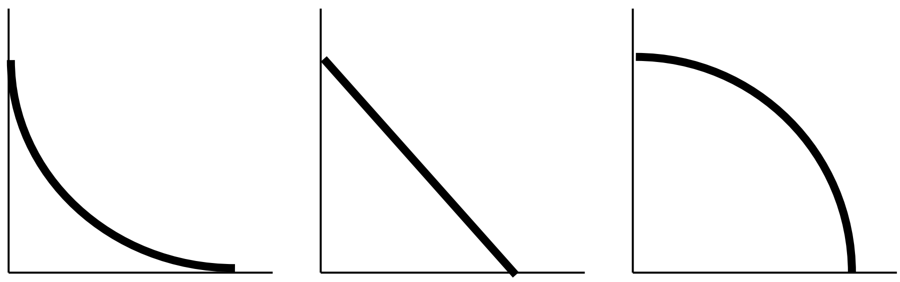


What distance will the person cover before reaching the barrel?

7. A cat on a ladder

Imagine a cat sitting half way up a ladder that is placed almost flush with a wall. If the base of the ladder is pushed fully up against the wall, the ladder and cat are most likely going to fall away from the wall (i.e. the top of the ladder falls away from the wall).

Part 1: If the cat stays on the ladder (not likely perhaps?) what will the trajectory of the cat be? A, B or C?



A B C

Part 2: Which of the above options represents the cat's trajectory if instead of the top of the ladder falling outwards, the base is pulled away? A, B or C?

8. Sailing

A yacht returns from a trip around the world. Different parts of the yacht have covered different distances. Which part of the yacht has covered the longest distance?

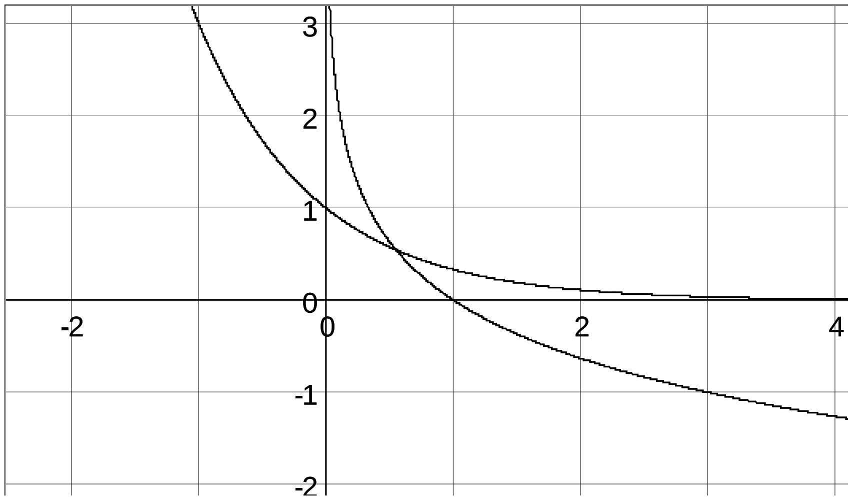
9. Encircling the Earth

Imagine a rope lying around the Earth's equator without any bends (ignore mountains and deep-sea trenches). The rope is lengthened by 20 metres and the circle is formed again. Estimate how high approximately the rope will be above the Earth:

A) 3 mm B) 3 cm C) 3 m?

10. A tricky equation

To check the number of solutions to the equation $\log_{\frac{1}{16}} x = \left(\frac{1}{16}\right)^x$ one can sketch the graphs of two inverse functions $y = \log_{\frac{1}{16}} x$ and $y = \left(\frac{1}{16}\right)^x$.



From the graphs we can see that there is one intersection point and therefore one solution to the equation $\log_{\frac{1}{16}} x = \left(\frac{1}{16}\right)^x$. But it is easy to check by substitution that both $x = \frac{1}{2}$ and $x = \frac{1}{4}$ satisfy the equation. So how many solutions does the equation have?

11. An alternative product rule

The derivative of the product of two differentiable functions is the product of their derivatives: $(uv)' = u'v'$. In which cases is this 'rule' true?

12. A 'strange' integral

Evaluate the following integral $\int \frac{dx}{dx}$.

13. Missing information?

At first glance it appears there is not enough information to solve the following problem: A circular hole 16 cm long is drilled through the centre of a metal sphere. Find the volume of the remaining part of the sphere.

14. A paint shortage

To paint the area bounded by the curve $y = \frac{1}{x}$, the x-axis and the line $x = 1$ is impossible. There is not enough paint in the world, because the area is infinite: $\int_1^{\infty} \frac{1}{x} dx = \lim_{b\to\infty}(\ln b - \ln 1) = \infty$.

However, one can rotate the area around the x-axis and the resulting solid of revolution would have a finite volume:

$\pi \int_1^{\infty} \frac{1}{x^2} dx = -\pi \lim_{b\to\infty}(\frac{1}{b} - \frac{1}{1}) = \pi$. This solid of revolution contains the area which is a cross-section of the solid. One can fill the solid with π cubic units of paint and thus cover the area with paint. Can you explain this paradox?

1.2 Sophisms

1. 1 = 0

We know that the limit of the sum of two sequences equals the sum of their limits, provided both limits exist. We also know that this is true for any number k of sequences in the sum.

Let us take n equal sequences $\frac{1}{n}$ and find the limit of their sum when $n \to \infty$:

$$\lim_{n\to\infty}\left(\frac{1}{n}+\frac{1}{n}+\ldots+\frac{1}{n}\right)=\lim_{n\to\infty}\frac{1}{n}+\lim_{n\to\infty}\frac{1}{n}+\ldots+\lim_{n\to\infty}\frac{1}{n}=0+0+\ldots+0=0.$$

On the other hand, the sum $\underbrace{\frac{1}{n}+\frac{1}{n}+\ldots+\frac{1}{n}}_{n}$ is equal to $n\times\frac{1}{n}=1$.

So we receive 1 = 0.

2. 1 = -1

   Let us find two limits of the same function:

   a) $\lim_{x\to\infty}\lim_{y\to\infty}\frac{x+y}{x-y}=\lim_{x\to\infty}\lim_{y\to\infty}\frac{\frac{x}{y}+1}{\frac{x}{y}-1}=\lim_{x\to\infty}(-1)=-1$.

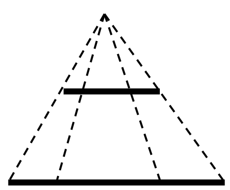
   b) $\lim_{y\to\infty}\lim_{x\to\infty}\frac{x+y}{x-y}=\lim_{y\to\infty}\lim_{x\to\infty}\frac{1+\frac{y}{x}}{1-\frac{y}{x}}=\lim_{y\to\infty}1=1$.

   Since $\lim_{x\to\infty}\lim_{y\to\infty}\frac{x+y}{x-y}=\lim_{y\to\infty}\lim_{x\to\infty}\frac{x+y}{x-y}$, the results from a) and b) must be equal.

   Therefore we conclude that 1 = -1.

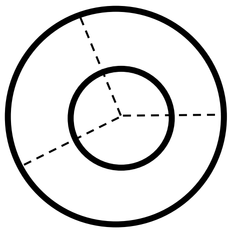
3. 1 = 2

   a) Take two line segments of length 1 unit and 2 units and establish a one-to-one correspondence between their points as shown on the diagram below:



The number of points on the line segment of length 1 unit is the same as the number of points on the line segment of length 2 units, meaning 1 = 2.

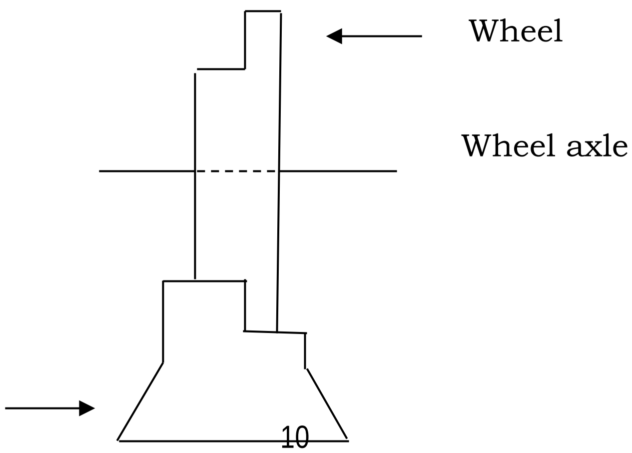
b) Take two circles of radius 1 unit and 2 units and establish a one-to-one correspondence between their points as shown on the diagram below:



The number of points on the circumference of the inner circle is the same as the number of points on the circumference of the outer circle, so we can conclude that 1 = 2.

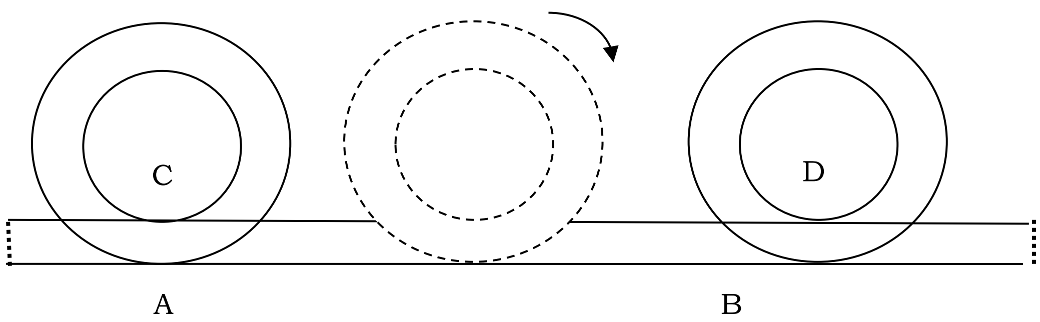
4. $R = r$

Two wheels of *different* radius are attached to each other and put on the same axis. Both wheels are on a rail (see the diagrams on the next page). After one rotation the large wheel with radius R covers the distance AB which is equal to the length of its circumference $2\pi R$. The small wheel with radius r covers the distance CD which is equal to the length of its circumference $2\pi r$. It is clear that AB = CD, therefore $2\pi R = 2\pi r$ and $R = r$.



Rail

Cross-section showing the wheel and the rail.



5. 2 > 3

   We start from the true inequality:

   $$\frac{1}{4} > \frac{1}{8} \text{ or } \left(\frac{1}{2}\right)^2 > \left(\frac{1}{2}\right)^3$$

   Taking natural logs of both sides:

   $$\ln\left(\frac{1}{2}\right)^2 > \ln\left(\frac{1}{2}\right)^3$$

   Applying the power rule of logs:

   $$2\ln\left(\frac{1}{2}\right) > 3\ln\left(\frac{1}{2}\right)$$

   Dividing both sides by $\ln\left(\frac{1}{2}\right)$:

   2 > 3.

6. 2 > 3

   We start from the true inequality:

   $$\frac{1}{4} > \frac{1}{8} \text{ or } \left(\frac{1}{2}\right)^2 > \left(\frac{1}{2}\right)^3$$

Taking logs with the base $\frac{1}{2}$ of both sides:

$$\log_{\frac{1}{2}}\left(\frac{1}{2}\right)^2 > \log_{\frac{1}{2}}\left(\frac{1}{2}\right)^3$$

Applying the power rule of logs:

$$2\log_{\frac{1}{2}}\left(\frac{1}{2}\right) > 3\log_{\frac{1}{2}}\left(\frac{1}{2}\right)$$

Since $\log_{\frac{1}{2}}\left(\frac{1}{2}\right) = 1$ we obtain: 2 > 3.

7. $\frac{1}{4} > \frac{1}{2}$

   We start from the true equality:

   $$\frac{1}{2} = \frac{1}{2}$$

   Taking natural logs of both sides:

   $$\ln\frac{1}{2} = \ln\frac{1}{2}$$

   Doubling the left hand side we obtain the inequality:

   $$2\ln\frac{1}{2} > \ln\frac{1}{2}$$

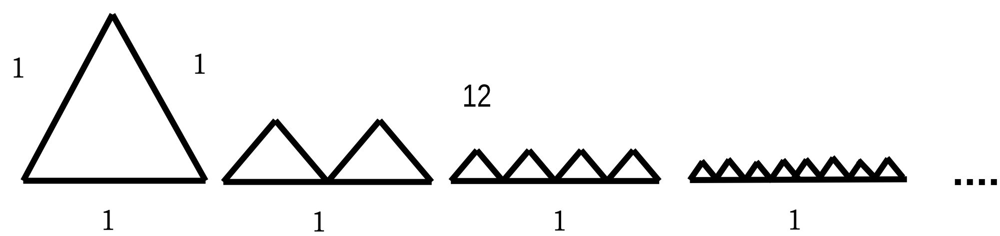
   Applying the power rule of logs:

   $$\ln\left(\frac{1}{2}\right)^2 > \ln\left(\frac{1}{2}\right)$$

   Since $y = \ln x$ is an increasing function $\left(\frac{1}{2}\right)^2 > \frac{1}{2}$ or $\frac{1}{4} > \frac{1}{2}$.

8. 2 = 1

   Let us take an equilateral triangle with sides of 1 unit. Divide the upper sides by 2 and transform them into a zig-zagging, segmented line as shown on the diagram below:



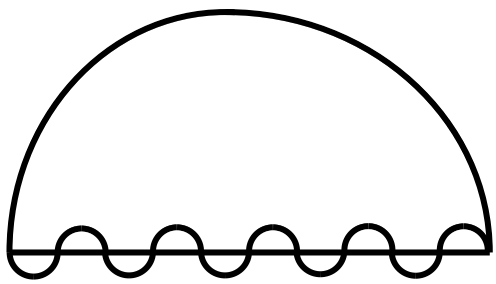
a) The length of this segment line is 2 units because it is constructed from two sides of 1 unit each. We continue halving the triangle sides infinitely many times. At *any* step the length of the segment line equals 2 units.

b) On the other hand from the diagram we can see that with more steps, the segment line gets closer and closer to the base of the triangle which has length 1 unit. That is $\lim_{n\to\infty} S_n = 1$, where S_n is the length of the segment line at step *n*.

Comparing a) and b) we conclude that 2 = 1.

9. $\pi = 2$

Let us take a semicircle with diameter *d*. We divide its diameter into *n* equal parts and on each part construct semicircles of diameter $\frac{d}{n}$ as shown on the following diagram:



a) The arc length of each small semicircle is $\frac{\pi d}{2n}$. The total length L_n of n semicircles is $L_n = \frac{\pi d}{2n} \times n = \frac{\pi d}{2}$. Therefore the limit of L_n when $n \to \infty$ is:

$\lim_{n \to \infty} L_n = \lim_{n \to \infty} \frac{\pi d}{2} = \frac{\pi d}{2}$.

b) From the diagram we can see that when n increases, the curve consisting of n small semicircles gets closer to the diameter, which has length d. That is $\lim_{n \to \infty} L_n = d$.

Comparing a) and b) we see that $\frac{\pi d}{2} = d$ and conclude that $\pi = 2$.

10. 1,000,000 ≈ 2,000,000

If we add 1 to a big number the result would be approximately equal to the original number. Let us take 1,000,000 and add 1 to it.
That is 1,000,000 ≈ 1,000,001.

Similarly 1,000,001 ≈ 1,000,002.

And 1,000,002 ≈ 1,000,003.

And so on…

1,999,999 ≈ 2,000,000.

Multiplying the left-hand sides and the right-hand sides of the above equalities we receive:

$1{,}000{,}000 \times 1{,}000{,}001 \times \ldots \times 1{,}999{,}999 \approx 1{,}000{,}001 \times 1{,}000{,}002 \times \ldots \times 2{,}000{,}000.$

Dividing both sides by $1{,}000{,}001 \times \ldots \times 1{,}999{,}999$ we conclude that

$1{,}000{,}000 \approx 2{,}000{,}000.$

11. 1 = -1

Since $\sqrt{a \times b} = \sqrt{a} \times \sqrt{b}$, it follows that

$1=\sqrt{1}=\sqrt{(-1)\times(-1)}=\sqrt{-1}\times\sqrt{-1}=i\times i=i^2=-1.$

12. 2 = -2

Two students were discussing square-roots with their teacher.

The first student said: “A square root of 4 is -2”.

The second student was sceptical, and wrote down $\sqrt{4}=2$.

Their teacher commented: “You are both right”.

The teacher was correct, so 2 = -2.

13. 2 = 1

Let us find the equation of a slant (or oblique) asymptote of the function $y=\dfrac{x^2+x+4}{x-1}$ using two different methods.

a) By performing long division: $\dfrac{x^2+x+4}{x-1}=x+2+\dfrac{6}{x-1}$. The last term, $\dfrac{6}{x-1}$ tends to zero as $x\to\infty$. Therefore as $x\to\infty$ the function approaches the straight line $y=x+2$, which is its slant asymptote.

b) Dividing both numerator and denominator by x we receive:

$\dfrac{x^2+x+4}{x-1}=\dfrac{x+1+\frac{4}{x}}{1-\frac{1}{x}}$. Both $\dfrac{4}{x}$ and $\dfrac{1}{x}$ tend to zero as $x\to\infty$. Therefore as $x\to\infty$ the function approaches the straight line $y=x+1$ which is its slant asymptote.

The function $y=\dfrac{x^2+x+4}{x-1}$ has only one slant asymptote. Therefore from a) and b) it follows that $x+2=x+1$.

Cancelling x we receive 2 = 1.

14. 1 = C, where C is *any* real number

Let us apply the substitution method to find the indefinite integral

$\int \sin x \cos x dx$ using two different methods:

a) $\int \sin x \cos x dx = \begin{bmatrix} u = \sin x \\ du = \cos x dx \end{bmatrix} = \int u du = \frac{u^2}{2} + C_1 = \frac{\sin^2 x}{2} + C_1$

b) $\int \sin x \cos x dx = \begin{bmatrix} u = \cos x \\ du = -\sin x dx \end{bmatrix} = -\int u du = -\frac{u^2}{2} + C_2 = -\frac{\cos^2 x}{2} + C_2$,

where C_1 and C_2 are arbitrary constants. Equating the right hand sides in a) and b) we obtain

$\frac{\sin^2 x}{2} + C_1 = -\frac{\cos^2 x}{2} + C_2$.

Multiplying the above equation by 2 and simplifying we receive

$\sin^2 x + \cos^2 x = 2C_2 - 2C_1$ or $\sin^2 x + \cos^2 x = C$ since the difference of two *arbitrary* constants is an arbitrary constant. On the other hand we know the trigonometric identity $\sin^2 x + \cos^2 x = 1$. Therefore 1 = C.

15. 1 = 0

Let us find the indefinite integral $\int \frac{1}{x} dx$ using the formula for integration by parts $\int u dv = uv - \int v du$:

$$\int \frac{1}{x} dx = \begin{bmatrix} u = \frac{1}{x} & du = -\frac{1}{x^2} \\ dv = dx & v = x \end{bmatrix} = \left(\frac{1}{x}\right)x - \int x\left(-\frac{1}{x^2}\right)dx = 1 + \int \frac{1}{x} dx.$$

That is, $\int \frac{1}{x} dx = 1 + \int \frac{1}{x} dx$.

Subtracting the same expression $\int \frac{1}{x} dx$ from both sides we receive 0 = 1.

16. Division by zero

Let us find the indefinite integral $\int \frac{dx}{2x+1}$ by the formula $\int \frac{f'(x)dx}{f(x)} = \ln|f(x)| + C$ using two different methods:

a) $\int \frac{dx}{2x+1} = \frac{1}{2}\int \frac{dx}{x+\frac{1}{2}} = \frac{1}{2}\ln\left|x+\frac{1}{2}\right| + C_1$.

b) $\int \frac{dx}{2x+1} = \frac{1}{2}\int \frac{2dx}{2x+1} = \frac{1}{2}\ln|2x+1| + C_2$.

Equating the right hand sides in a) and b) we obtain

$$\frac{1}{2}\ln\left|x+\frac{1}{2}\right| + C_1 = \frac{1}{2}\ln|2x+1| + C_2$$

Since C_1 and C_2 are arbitrary constants that can take *any* values, let $C_1 = C_2 = 0$. Then $\frac{1}{2}\ln\left|x+\frac{1}{2}\right| = \frac{1}{2}\ln|2x+1|$.

Solving for x we receive $x+\frac{1}{2} = 2x+1,\; x = -\frac{1}{2}$.

Substituting this value of x into the original integral gives zero in the denominator, so division by zero is possible!

17. π is not defined

Let us find the limit $\lim\limits_{x\to\infty}\frac{\pi x+\sin x}{x+\sin x}$ using two different methods.

a) $\lim\limits_{x\to\infty}\frac{\pi x+\sin x}{x+\sin x} = \lim\limits_{x\to\infty}\frac{\pi+\frac{\sin x}{x}}{1+\frac{\sin x}{x}} = \pi$.

b) Since both numerator and denominator are differentiable we can use the well known rule $\lim\limits_{x\to\infty}\frac{f(x)}{g(x)} = \left[\frac{\infty}{\infty}\right] = \lim\limits_{x\to\infty}\frac{f'(x)}{g'(x)}$ which gives us:

$\lim\limits_{x\to\infty}\frac{\pi x+\sin x}{x+\sin x} = \left[\frac{\infty}{\infty}\right] = \lim\limits_{x\to\infty}\frac{\pi+\cos x}{1+\cos x}$, which is undefined.

Comparing the results in a) and b) we conclude that π is not defined.

18. 0 = C, where C is *any* real number

We know the property of an indefinite integral:

$\int kf(x)dx = k\int f(x)dx$ where k is a constant.

Let us apply this property for k = 0.

a) The left hand side of the above equality is $\int 0 f(x)dx = \int 0dx = C$, where C is an arbitrary constant.

a) The right hand side is $0\int f(x)dx = 0$.

Comparing a) and b) we conclude that 0 = C, where C is *any* real number.

19. 1 = 2

Let us find the volume of the solid of revolution produced by rotating the hyperbola $y^2 = x^2 - 1$ about the x-axis on the interval [-2, 2] using two different methods.

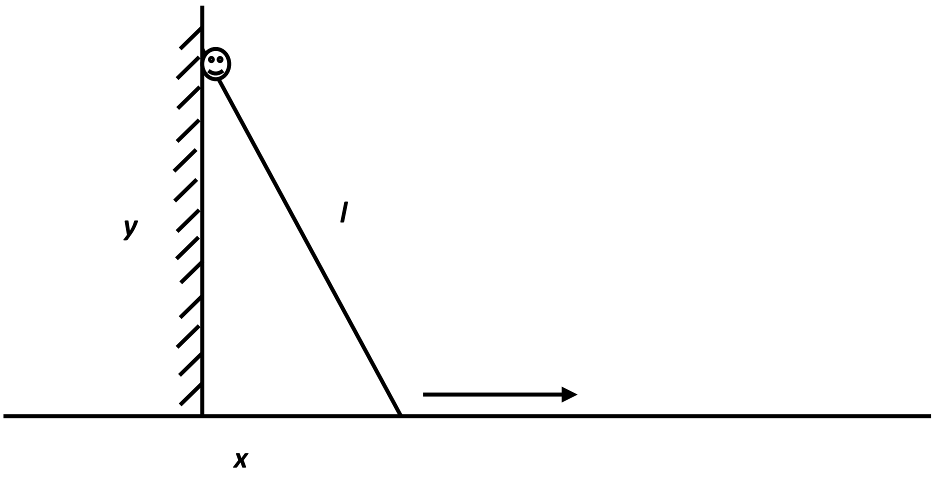
a) $V = \pi\int_{-2}^{2} y^2 dx = \pi\int_{-2}^{2} (x^2 - 1)dx = \pi(\frac{x^3}{3} - x)\Big|_{-2}^{2} = \frac{4}{3}\pi$ (cubic units).

b) Since the hyperbola is symmetrical about the y-axis we can find the volume of a half of the solid of revolution, say on the right from the y-axis and then multiply it by 2. Obviously the point (1,0) is a vertex to the right of the origin and the right branch of the hyperbola is to the right of the vertex (1,0). Therefore the volume of the right half is $V_1 = \pi\int_{1}^{2} y^2 dx = \pi\int_{1}^{2} (x^2 - 1)dx = \pi(\frac{x^3}{3} - x)\Big|_{-2}^{2} = \frac{4}{3}\pi$ (cubic units) and the total volume $V = 2V_1 = \frac{8}{3}\pi$ (cubic units).

Comparing a) and b) we obtain $\frac{4}{3}\pi = \frac{8}{3}\pi$ or 1 = 2.

20. An infinitely fast fall

Imagine a cat sitting on the top of a ladder leaning against a wall. The bottom of the ladder is pulled away from the wall horizontally at a uniform rate. The cat speeds up, until it's falling infinitely fast. The 'proof' is below.



By the Pythagoras Theorem $y = \sqrt{l^2 - x^2}$, where $x = x(t)$, $y = y(t)$ are the horizontal and vertical distances from the ends of the ladder to the corner at time t. Differentiation of both sides with respect to t gives us $y' = -\frac{xx'}{\sqrt{l^2 - x^2}}$. Since the ladder is pulled *uniformly* x' is a constant. Let us find the limit of y' when x approaches l: $\lim_{x \to l} y' = \lim_{x \to l} \left(-\frac{xx'}{\sqrt{l^2 - x^2}} \right) = -\infty$.

When the bottom of the ladder is pulled away by the distance l from the wall, the cat falls infinitely fast.

Chapter 2

Puzzles in Physics and Astronomy

1. Eratosthenes' Problem

The ancient philosopher Eratosthenes of Cyrene (276–194 B.C.) lived in Alexandria and worked as a librarian in the famous Alexandria Library. To work out the size of the Earth, Eratosthenes used two well-known facts:

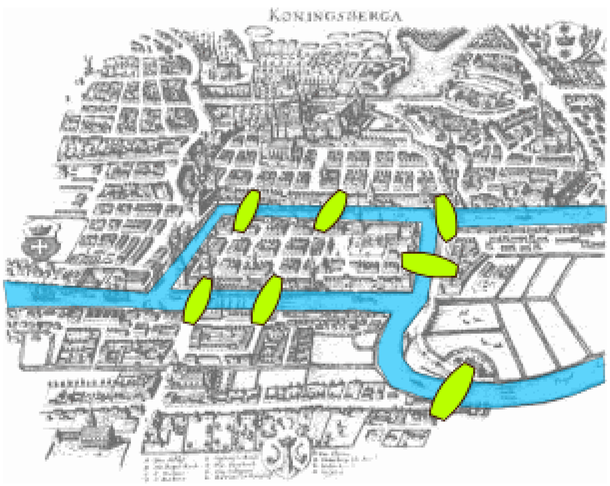
1. On Midsummer's Day, the Sun's rays reach the bottom of the deepest well in Syene (now Aswan on the River Nile in Egypt).
2. Syene is 5000 stadia (an ancient Greek measurement of length), which is 800 kilometres (using today's metric system), south of Alexandria.

Eratosthenes also used his own measurements of the height of the Sun at noon on Midsummer's Day, which he performed on the roof of Alexandria Library. The Sun happened to be $1/_{50}$ part of the circle, or 7.2° away from the zenith (the point in the sky directly above his head).

Living now, that is 23 centuries after Eratosthenes, can you solve the problem of the size of the Earth solely on the basis of this data?

2. Euler's Problem

Leonard Euler (1707–1783), one of the greatest mathematicians, visited Königsberg, Germany. At that time there were 7 bridges in Königsberg that connected the banks of the local river and two islands. Euler posed the following question: Is it possible to walk across all 7 bridges (shown in green in the lithograph), walking across each bridge just once?



3. Reflection in a Mirror

Look in the mirror and touch your right cheek with your finger. Your reflection shows the finger touching the left cheek. If you wave your right hand, your reflection will wave the left hand. This is because the mirror reverses the left and right in our reflected image. We understand that this is an optical effect, and even without deep knowledge of Physics we can accept it as a law of optics.

But then an obvious question arises: If a mirror reverses left and right, why doesn't it also reverse top and bottom? For instance, if you face a mirror and pat the top of your head, your image will do the same.

Here are most frequent answers to this question. Which one is correct?

a. This effect is due to the fact that our eyes are aligned horizontally on our faces.
b. It is gravity which is at work. (Doesn't that define "up" and "down" for all?)
c. Our brains are divided into left and right halves. The left half of the brain controls the right half of the body while the right half controls the left. They have very different functions in our mind. Our speech comes from the left side while visual recognition is in the right. Although we look symmetrical externally, our psyche is very asymmetrical. Could that have anything to do with it?
d. None of the above is right.

4. 10 Types of People

There are 10 types of people in the world: those who understand binary code and those who don't. Does this make sense to you?

5. A "stupid" question about bytes and bits

One byte consists of 8 bits. Why 8? Why not 7 or 9?

6. “Ctrl+Alt+Del”

Every day your PC asks you to type “Ctrl+Alt+Del” key combination to reboot your computer. Why “Ctrl+Alt+Del”?

7. Keyboard Puzzles

Where is the "Any" key on your keyboard?

8. How many Fridays are there in February?

What is the greatest and least number of Fridays in February?

9. Add a Metre

The Earth revolves around the Sun at a distance of 150,000,000 km. Suppose we add one metre to this distance. How much longer would be the Earth's path around the Sun and how much longer the year, provided the velocity of the Earth's orbital motion remained the same?

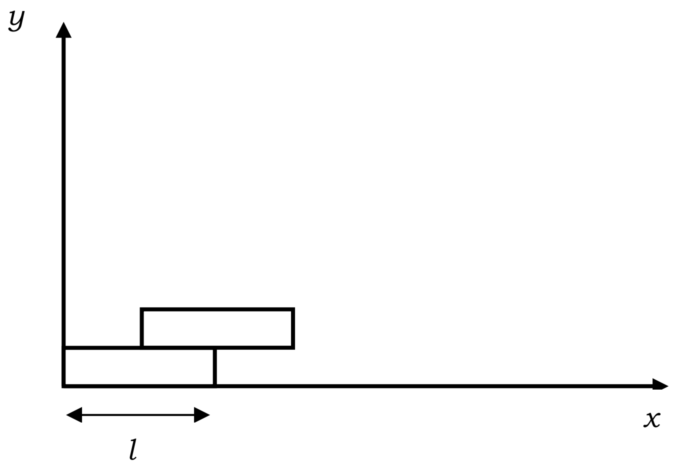
Solutions for Chapter 1.1 Paradoxes

1. Laying bricks

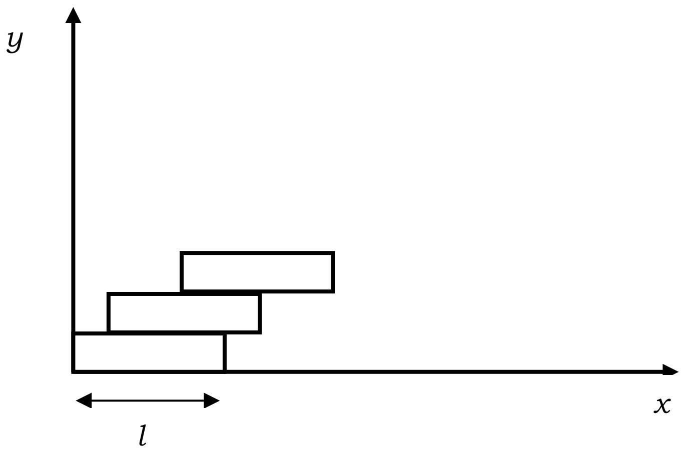
   The top brick can be infinitely far from the bottom brick! The x-coordinate of the position of the centre of mass of a system of n objects with masses m_1, m_2,…, m_n is defined by the formula:

$$x_0 = \frac{m_1 x_1 + m_2 x_2 + \ldots + m_n x_n}{m_1 + m_2 + \ldots + m_n}.$$

Let us consider two bricks. For the upper brick not to fall from the lower brick the perpendicular distance from the centre of mass of the upper brick should not be beyond the right edge of the lower brick. That is, the maximum value of the x coordinate of the centre of mass of the upper brick is l: $x_0 = l$. So the maximum shift is: $\Delta x_1 = \frac{l}{2}$.



Let us consider three bricks.



For the top brick we have the maximum possible shift: $\Delta x_1 = \frac{l}{2}$. Let us find the maximum possible shift for the middle brick. Again, the perpendicular distance from the centre of mass of the system of the middle and top bricks should not extend past the right edge of the lower brick. In other words, the maximum value of the x coordinate of the centre of mass of the system of the middle and the top bricks is l: $x_0 = l$. Expressing x_0 for the system of the middle and the top bricks from (1) we obtain:

$$\frac{m(\Delta x_2 + \frac{l}{2}) + m(\Delta x_2 + \frac{l}{2} + \frac{l}{2})}{2m} = l .$$

From here $\Delta x_2 = \frac{l}{4}$.

In a similar way we can obtain: $\Delta x_3 = \frac{l}{6}, \Delta x_4 = \frac{l}{8}, \dots , \Delta x_n = \frac{l}{2n}$.

Adding all shifts we receive: $\Delta x_1 + \Delta x_2 + \dots + \Delta x_n = \frac{l}{2}(1 + \frac{1}{2} + \dots + \frac{1}{n})$.

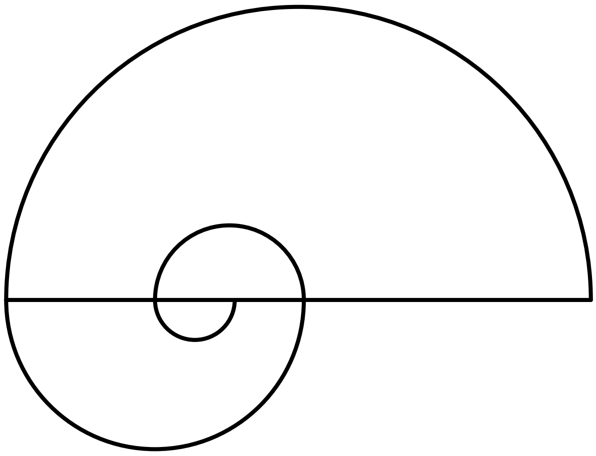
When $n \to \infty$ the sum in the brackets tends to infinity. This means that the maximum possible shift of the top brick with respect to the bottom brick can be made as large as we want.

Comment: In practice it is of course impossible. Starting from a certain value of n we will not be able to make shifts of the length $\frac{l}{2n}$ as they will be too small to perform.

2. Spiral curves

a) Let us construct a spiral curve of a finite length.

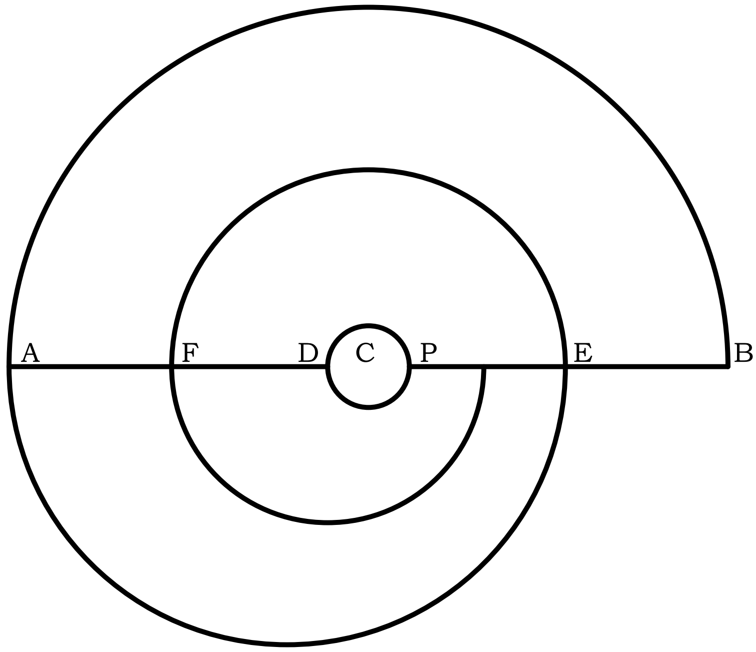
Draw a line segment of length d. Draw a semicircle with diameter d on one side of the line segment. Then on the other side of the line segment draw a semicircle of diameter $d/2$. Then on the other side draw a semicircle of diameter $d/4$, and so on.



The length of the curve is:

$$\pi\frac{d}{2}+\pi\frac{d}{4}+\pi\frac{d}{8}+\ldots=\pi d(\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots)=\pi d\ .$$

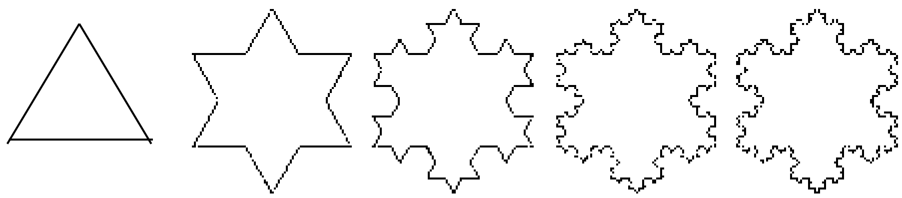
b) Let us construct a spiral curve of infinite length. Draw a line segment AB of length d with midpoint P. Draw a circle with the centre C on the line segment at distance a from P. On one side of the line segment draw a semicircle of the diameter d. On the other side of the line segment draw a semicircle of the diameter AE, where point E is the midpoint of PB. Then on the other side of the line segment draw a semicircle of the diameter EF, where point F is the midpoint of AD and so on (see the following diagram).



The curve has infinitely many rotations around point C and each rotation has a length bigger than the circumference $2\pi a$, so the length of the curve is infinite.

3. A tricky curve

One example of such a curve is the famous Koch snowflake. We start with an equilateral triangle and build the line segments on each side according to a simple rule: at every step each line segment is divided into 3 equal parts, then the process is repeated infinitely many times. The resulting curve is called the Koch curve or Koch snowflake and is an example of a fractal. The first four iterations are shown below:



The initial triangle and all consecutive stars and snowflakes are located between the circumferences inscribed into the triangle and circumscribed around it. Both circumferences have finite lengths. If the perimeter of the initial triangle is 1 unit, then the perimeter of the star in the first iteration is $12\times\frac{1}{9}=\frac{4}{3}$ units. The perimeter of the snowflake in the second iteration is $48\times\frac{1}{27}=\frac{16}{9}=\left(\frac{4}{3}\right)^2$ units.

The perimeter of the snowflake in the n^{th} iteration is $\left(\frac{4}{3}\right)^n$ units.

As $n\to\infty$ the perimeter of the snowflake tends to infinity. The Koch curve has an infinite length but bounds a finite area, which is between the area of the circle inscribed into the initial triangle and the area of the circle circumscribed around it.

4. A tricky area

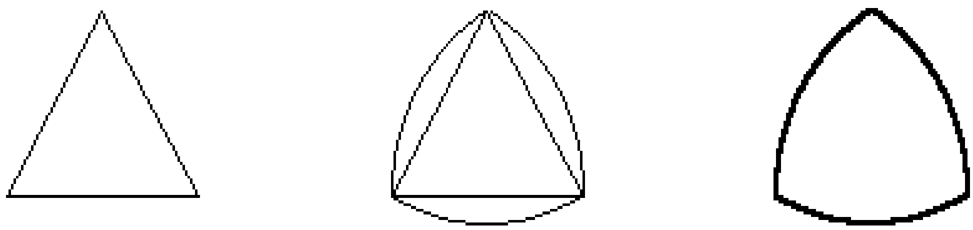
Although at every step the remaining area is 8 times bigger than the area removed, after infinitely many steps the remaining area will be zero and the area removed will be 1 square unit. Let us show this. After the first step the remaining area equals $1-\frac{1}{9}=\frac{8}{9}$.

After the second step the remaining area is $\frac{8}{9}-8\times\frac{1}{81}=\frac{64}{81}=\left(\frac{8}{9}\right)^2$.

Similarly, after the n^{th} step the remaining area is $\left(\frac{8}{9}\right)^n$ and if n tends to infinity this area tends to zero. This figure is called the Sierpinski carpet and is another example of a fractal.

5. A tricky shape

No, the student was not right. A figure can be of constant diameter yet not be a circle. As an example, consider the following curve. In an equilateral triangle draw circular arcs with the radius equal to the side of the triangle from each vertex. The resulting figure is a curved triangle, which is called the Reuleaux triangle (see the diagram below). One of its properties is that it has a constant diameter. When it rolls on a horizontal surface its center moves along a sine curve with ups and downs (unlike a circle whose center does not move up and down – only along a straight horizontal line). For this reason it is not practical to use it as a wheel, but it does have practical applications. It is used in the Wankel rotary engine, and in some countries manhole covers are shaped like the Reuleaux triangle.

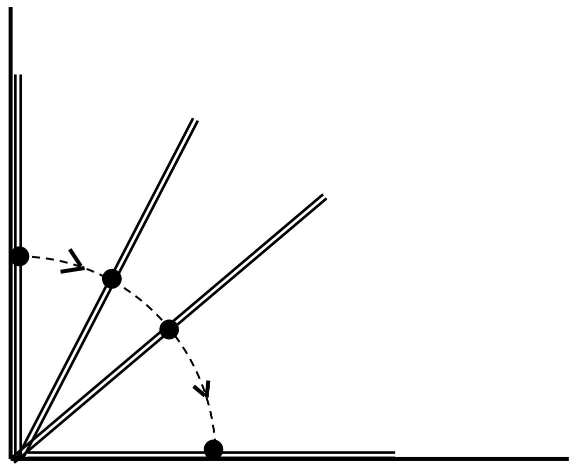


6. Rolling a barrel

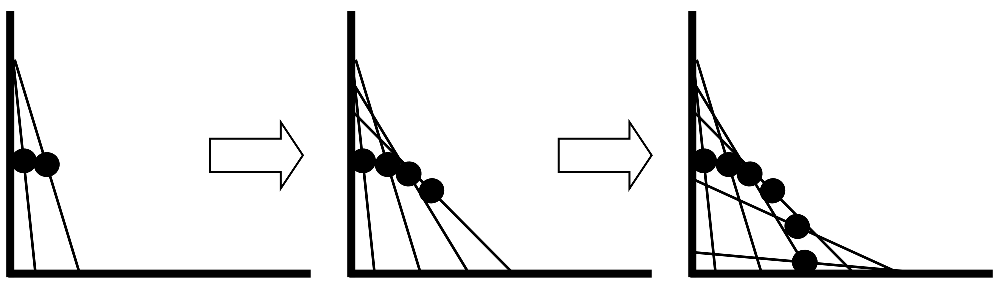
The barrel rolls as long as the person continues walking. The velocity of the point on the top of the barrel equals the velocity of the walking person and is twice the velocity of the axis of the barrel, so the person will cover 6 m by the time he reaches the barrel.

7. A cat on a ladder

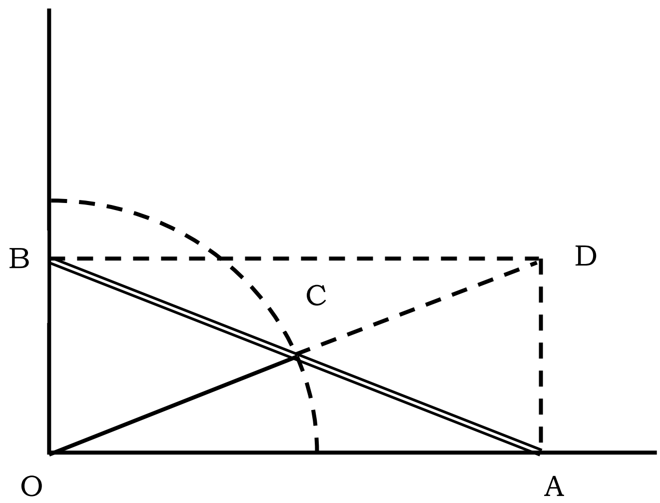
Part 1. Most people are confident that C is the answer to Part 1. Without much difficulty, one can imagine the ladder rotating about a central point, i.e. where the base of the ladder touches the wall. An arc is the result and in this case represents a quarter of the circumference of a circle.



Part 2. However, Part 2 isn't so easy! Many people conclude that A is the correct answer. It sounds reasonable that as the ladder slides outwards away from the wall that it would appear to drop quickly, then level out as it approaches the horizontal. Surprisingly, the answer to this problem is also C. Try it out by making a model (see the sequential sketches below). With a paper ladder, with a point marking half way, slowly slide the ladder down and away from the wall. After each small amount of movement, put a dot on the page at the place where the centre of the ladder lies. Note that as the ladder approaches the horizontal, further lateral movement is minimal.



Here is a simple proof:



Let AB be the ladder. Point C (where the cat sits) is always at the same distance (half the length of the ladder) from the point O regardless of the position of the ladder. This comes from the fact that diagonals in a rectangle are the same and are divided in half by the point of their intersection.

You may well be very surprised to see that the trajectory is the same in both cases. Do not be alarmed however – in this case the intuition of many people fails. My colleague tried this test out with a class of 100 4th year engineering students in Australia, Germany, New Zealand and Norway. These young men and women, aged about 21, are expected to be able to quickly conceptualise shapes, dimensions, movements and forces. The students were given 40 seconds to find the answer. They were told that it was a mental exercise, with no calculations or drawings permitted. The results were startling, for although 74% of the students gave C, the correct answer to Part 1, 86% were wrong in Part 2. 14% and 34% gave B as the answer in Parts 1 and 2 respectively.

8. Sailing

The top of the yacht has covered the longest distance. The shape of the Earth is approximately spherical, so the top of the yacht has the longest radius compared to lower parts and therefore has the longest circumference.

9. Encircling the Earth

Approximately 3 m high. This is a surprising answer for many people. Let r be the radius of the Earth and R be the radius of the circle after adding 20 metres to the rope. The difference between the two circumferences is 20 m: $2\pi R - 2\pi r = 20$ or $2\pi(R-r) = 20.$ From here the difference between the two radii is $R - r \approx 3m.$ The answer does not depend on the original length of the rope.

10. A tricky equation

The rough sketch of the graphs is "too rough". Both functions are decreasing for all x in their domains but they are very close to both axes, and in fact have 3 intersection points. It can be shown that the equation has 3 solutions.

11. An alternative product rule

The 'rule' is true when:

a) both functions u and v are constants

b) one of the functions is zero – the other can be any function

c) one function can be any function and the other is a solution of the differential equation $u'v' = u'v + uv'$.

For example, if $u = x$ then $v' = v + xv'.$ Solving for v we obtain:

$$v'(1-x) = v$$

$$\frac{dv}{v} = \frac{dx}{1-x}$$

$$\ln|v| = -\ln|1-x| + \ln|c|$$

$$\ln|v| = \ln\left|\frac{c}{1-x}\right|$$

$$v = \frac{c}{1-x},$$

where c is an arbitrary constant. In particular $v = \dfrac{1}{1-x}.$

As a result, there are infinitely many pairs of functions for which the 'rule' is true.

12. A 'strange' integral

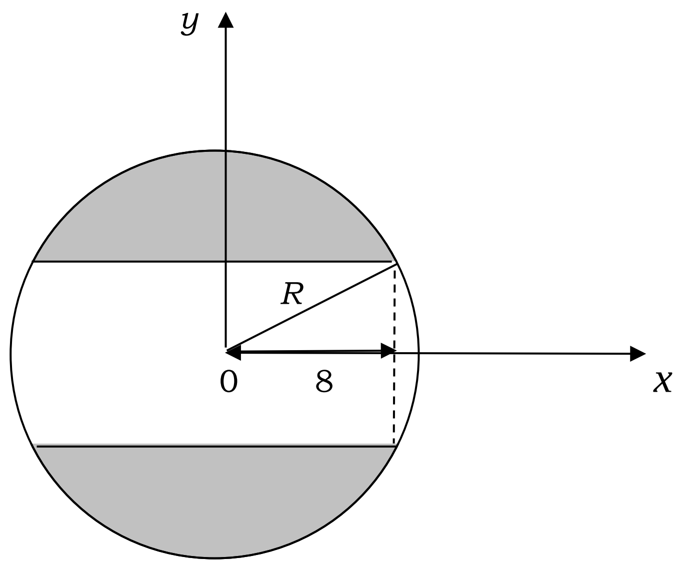
Obviously the symbol "*dx*" in the numerator is part of the notation of the operation of integration. The symbol "*dx*" in the denominator is not part of that notation. Since *x* is an independent variable then "*dx*" in the denominator means the product of a constant *d* and *x*. So the solution is: $\int \frac{dx}{dx} = \frac{1}{d}\int \frac{dx}{x} = \frac{1}{d}\ln|x| + c.$

(Supplied by Vitali Babakov, Auckland University of Technology, New Zealand)

13. Missing information?

Let *R* be the radius of the sphere. The radius of the drilled hole is then $r = \sqrt{R^2 - 64}$. The volume that remains once the hole has been drilled equals the volume of the solid of revolution when the shaded area rotates around the *x*-axis (see the following diagram):

$$V = 2\pi\int_0^8 (R^2 - x^2 - r^2)dx = 2\pi\int_0^8 (R^2 - r^2 - x^2)dx = 2\pi\int_0^8 (64 - x^2)dx = 2145\,cm^3.$$



14. A paint shortage

This is one of the most famous paradoxes to involve areas and volumes. The solid of revolution is called Torricelli's trumpet or Gabriel's Horn. The general approach to explain paradoxes that involve infinity and physical objects (like paint in this case) is to differentiate the "mathematical" universe from the "physical" universe. Infinity is a pure mathematical concept or idea and we cannot apply it to real, finite objects. From a mathematical point of view one "abstract" drop of paint is enough to cover any area, no matter how big. One just needs to make the thickness of the cover very thin, and infinitely thin if you want to cover an infinite area. Consider an easier example. You have 1 drop of paint which has the volume of 1 cubic unit. You need to cover a square plate of the size x by x units. Then the (uniform) thickness of the cover will be $\frac{1}{x^2}$ units. If say x = 100 cm then the thickness of the cover is $\frac{1}{10000}$ cm. If $x \to \infty$ then the area $x^2 \to \infty$ and the thickness $\frac{1}{x^2} \to 0$. But at any stage the volume is $x^2 \times \frac{1}{x^2} = 1$ cubic cm. So mathematically you can cover any infinite area with any finite amount of paint, even with a single drop. In reality such infinite areas don't exist, nor can one make the cover infinitely thin.

Solutions for Chapter 1.2 Sophisms

1. The property for evaluating the limit of the sum of sequences is true for any *finite* number k of sequences (provided that their limits exist). For n sequences in the sum where $n \to \infty$ this is not true.

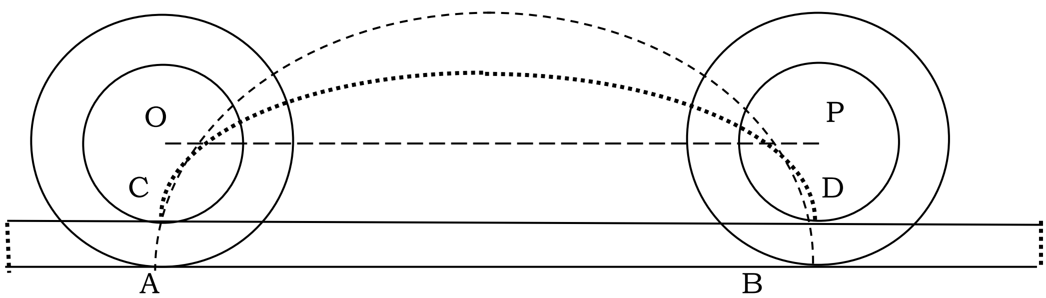
2. The order of the limits is important. Although the function is the same, changing the order can give different results. In general $\lim_{x\to a}\lim_{y\to b} f(x,y) \neq \lim_{y\to b}\lim_{x\to a} f(x,y)$. For example, $\lim_{x\to 0}\lim_{y\to 0} y\sin\frac{1}{x} = 0$ whereas $\lim_{y\to 0}\lim_{x\to 0} y\sin\frac{1}{x}$ is not defined because $\lim_{x\to 0} y\sin\frac{1}{x}$ does not exist.

3. Because there are *infinitely* many points on any line segment and any circumference it doesn't make sense to talk about the *number* of points. The one-to-one correspondence does not mean that the lengths are the same. The rules for sets with infinitely many elements are different to the rules for sets with a finite number of elements. By definition, a set is infinite if there is a one-to-one correspondence between its elements and the elements of one of its proper subsets. One can establish a one-to-one correspondence between points of say a line segment of 1 cm length and points on the whole infinite number line. One can establish a one-to-one correspondence between points on a 1 cm line segment and points on a 1 cm by 1 cm square. One can even establish a one-to-one correspondence between points on a 1 cm line segment and points of any 3-dimentional figure, for example a sphere of the size of the Earth.

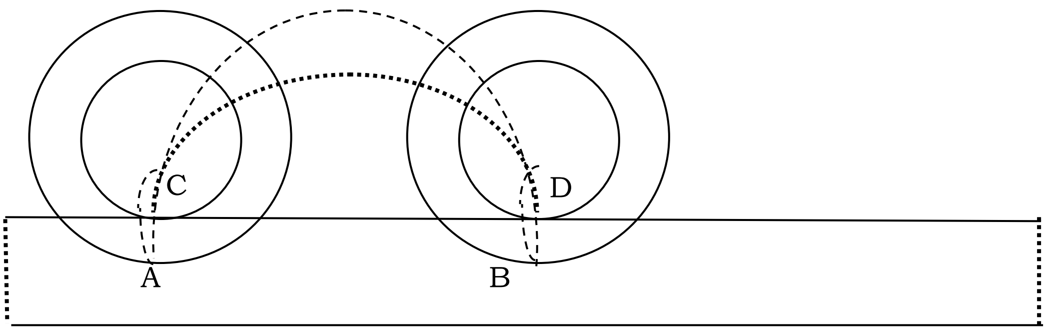
4. This sophism is known as Aristotle's Wheel. The great Greek philosopher Aristotle (384-322 BC) described it in his book "Problems of Mechanics". However, his explanation was not clear.

Later Galileo Galilei (1564-1642) gave his own explanation of the sophism. The mathematical essence of the sophism has finally been established only with use of the concept of one-to-one correspondence between sets (sets of equal power) discovered by Georg Cantor (1845-1918). The sophism is similar to sophism 9b).

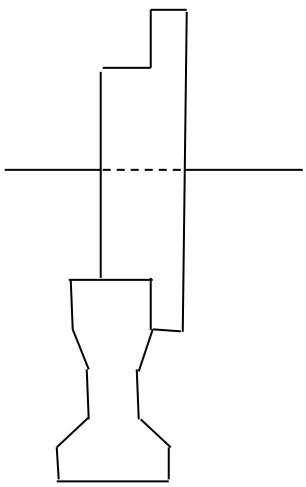
From a practical point of view it is impossible for both wheels to roll. Only the big wheel can roll, and when it does the small wheel both rotates and slides on the surface of the rail. When the big wheel makes one rotation and covers distance AB, point C moves to point D. Distance AB equals the circumference of the big wheel. Obviously AB = CD. But CD is bigger than the circumference of the small wheel, because apart from making one rotation the small wheel also slides on the surface of the rail. Point A on the big wheel traces a curve called a cycloid. Point C on the small wheel traces a curve which looks like a flattened cycloid. If the radius of the small wheel is very small, almost zero, then the trajectory of point C will be very close to the straight line OP. In this case the small wheel would be mostly sliding, because the distance CD would be much bigger than the length of the circumference of the small wheel. It is very unlikely that anything would ever be constructed in this way, because sliding of the small wheel will always create friction.



The small wheel can only roll if the big wheel doesn't contact the surface. Point C on the small wheel traces a cycloid and point A on the big wheel traces a curve that resembles an "unfinished circumference". If the radius of the small wheel is very small, almost zero, then the trajectory of point A will be very close to the circumference of the big wheel. For some time during the big wheel's revolution its lowest point will actually move in the opposite direction to that of the overall movement, and there will be a small loop on the bottom of its trajectory.



This is how physical railway wheels move. The small wheel rolls and the big wheel does not touch the surface. See the diagram below, which shows a cross-section through the wheel and the rail.



5. The mistake is in the last step. The number $\ln\frac{1}{2}$ is negative, therefore when dividing by it we need to change the sign of the inequality.

6. The mistake is in step 2. The function $y=\log_{\frac{1}{2}} x$ is decreasing and therefore from the inequality $\left(\frac{1}{2}\right)^2 > \left(\frac{1}{2}\right)^3$ it follows that $\log_{\frac{1}{2}}\left(\frac{1}{2}\right)^2 < \log_{\frac{1}{2}}\left(\frac{1}{2}\right)^3$.

7. The mistake is in step 3. The number $\ln\frac{1}{2}$ is negative, therefore when doubling the left hand side we obtain $2\ln\frac{1}{2} < \ln\frac{1}{2}$.

8. As *n* tends to infinity the length of each line segment approaches zero, but from the diagram we can see that the segment line tends to the triangle base (1). This is true but it doesn't mean that the *length* of the segment line tends to the *length* of the base. When *n* is increasing, the length of each line segment is getting smaller but the number of segments is getting bigger. At the first step line segment length is $\frac{1}{2}$, the number of the line segments is 4 so the length of the segment line is $4\times\frac{1}{2}=2$. At the second step the length of each line segment is $\frac{1}{4}$, the number of the line segments is 8 and therefore the length of the segment line is $8\times\frac{1}{4}=2$. At the n^{th} step the line segment length is $\frac{1}{2^n}$ and the number of the segments is 2^{n+1}, therefore the length of the segment line is $2^{n+1}\times\frac{1}{2^n}=2$. So when *n* tends to infinity the limit of the segment line's length is: $\lim\limits_{n\to\infty}(2^{n+1}\times\frac{1}{2^n}) = \lim\limits_{n\to\infty} 2 = 2$.

Note: When $n \to \infty$ the sequence of segment lines L_n tends to the segment line L of a finite length if the distance $d\ (L_n, L)$ tends to zero. Suppose $L_n \to L$. This doesn't mean that $\lim_{n\to\infty} l(L_n) = l(L)$, only that the inequality $\lim_{n\to\infty} l(L_n) \geq l(L)$ holds (provided that the limit $\lim_{n\to\infty} l(L_n)$ exists). In fact, the limit $\lim_{n\to\infty} l(L_n)$ can be *any* big number. In general, if curve A gets closer to curve B it doesn't mean that the *length* of curve A tends to the *length* of curve B.

9. The mistake is in b). As with the previous sophism we cannot rely on the problem's visual representation. When n increases the small semicircles get smaller, but their number increases and the total length of the curve equals $\frac{\pi d}{2}$ for *any n,* as was shown in a).

10. The rule for multiplying left and right sides of equalities is only valid for *exact* equalities, and is invalid for *approximate* equalities. Approximate equalities are actually inequalities. For example, $x \approx b$ with accuracy of 0.1 means that $b - 0.1 < x < b + 0.1$. In multiplying left hand sides and right hand sides of approximate equalities we actually multiply inequalities of the same nature. For example, for positive *a, b, c, d* from $a < b$ and $c < d$ it follows that $ac < bd$.

11. The property $\sqrt{a \times b} = \sqrt{a} \times \sqrt{b}$ is valid only for non-negative numbers a and b.

12. All three statements are true but the conclusion is certainly wrong. By definition the number 4 has two square roots, 2 and -2 (the result is 4 when either are squared). The first student mentioned just one of the square roots -2 ("*a* square root of 4 is -2", which is equivalent to saying "*one* square root of 4 is -2"). The second student gave the other (non-negative) square root of 4 because the symbol $\sqrt{\ }$ is used to represent only the non-negative square root. So the square roots of 4 are 2 and – 2 and the equality $\sqrt{4} = 2$ actually reads "the *non-negative* square root of 4 is 2", not "the square root of 4 is 2".

13. The mistake is in b). The technique in b) is used to evaluate the limit of a function when $x \to \infty$, but should not be used in the equation of the function. By definition, a straight line $y = mx + c$ is a slant asymptote to a function $y = f(x)$ if $\lim_{x\to\infty} |f(x) - (mx + c)| = 0$. From here it is easy to get the values of *m* and *c*: $m = \lim_{x\to\infty} \left|\frac{f(x)}{x}\right|$, $c = \lim_{x\to\infty} |f(x) - mx|$. Another way is to use long division.

14. In both sides of the equation $\frac{\sin^2 x}{2} + C_1 = -\frac{\cos^2 x}{2} + C_2$ we have the *same* infinite set of functions written in different forms. For each antiderivative of the left hand side with a certain value of C_1 we can find the same antiderivative of the right hand side with another value of C_2 and vice versa. Although being arbitrary, the constants C_1 and C_2 are *not independent* of each other. They are related by the formula $2C_2 - 2C_1 = 1$ that can be obtained with the use of the trigonometric identity $\sin^2 x + \cos^2 x = 1$ (for example substitute $(1 - \sin^2 x)$ instead of $\cos^2 x$ in the equation $\frac{\sin^2 x}{2} + C_1 = -\frac{\cos^2 x}{2} + C_2$). The difference of two arbitrary but *dependent* constants is not an arbitrary constant. In our case the difference $2C_2 - 2C_1$ equals 1.

15. This is similar to the previous sophism. In both sides of the equation $\int \frac{1}{x} dx = 1 + \int \frac{1}{x} dx$ we have the *same* infinite set of functions written in different forms: $\ln|x| + C_1 = 1 + \ln|x| + C_2$. For each antiderivative of the left hand side with a certain value of C_1 we can find the same antiderivative of the right hand side with another value of C_2 and vice versa. Although being arbitrary, the constants C_1 and C_2 are *not independent* of each other. They are related by the formula $C_1 = 1 + C_2$ which we obtain after subtracting the log function from both sides of the equation $\ln|x| + C_1 = 1 + \ln|x| + C_2$.

16. Although being arbitrary, the constants C_1 and C_2 are *not independent* of each other. They are related by the formula that can be obtained by applying the rules for logs:

$$\frac{1}{2}\ln\left|x+\frac{1}{2}\right|+C_1=\frac{1}{2}\ln|2x+1|+C_2$$

$$\frac{1}{2}\ln\left|x+\frac{1}{2}\right|+C_1=\frac{1}{2}\ln\left|2(x+\frac{1}{2})\right|+C_2$$

$$\frac{1}{2}\ln\left|x+\frac{1}{2}\right|+C_1=\frac{1}{2}\ln 2+\frac{1}{2}\ln\left|x+\frac{1}{2}\right|+C_2$$

$$C_1=\ln\sqrt{2}+C_2.$$

Separately C_1 and C_2 can equal zero, just not at the same time!

17. The mistake is in b). The "well known rule" has one more condition: "if the limit $\lim_{x\to\infty}\frac{f'(x)}{g'(x)}$ exists or equals $\pm\infty$". This is l'Hospital's Rule, and can't be applied here since $\lim_{x\to\infty}\frac{f'(x)}{g'(x)}$ does not exist.

18. The property $\int kf(x)dx=k\int f(x)dx$ is valid only for *non-zero* values of *k*. The same property of the *definite* integral is valid for any value of *k*.

19. The mistake is in a). We cannot integrate y^2 over the interval [-2, 2] to find the volume because the hyperbola does not exist on the interval (-1, 1), since x^2-1 is negative for $-1 < x < 1$.

20. The 'proof' assumes that the ladder maintains contact with the wall while being pulled. This model is simply not true. If all forces involved are considered it can be shown that at one stage the top of the ladder will loose contact and be pulled away from the wall. From that moment the relationship $y=\sqrt{l^2-x^2}$ is no longer true, since we don't have a right-angled triangle.

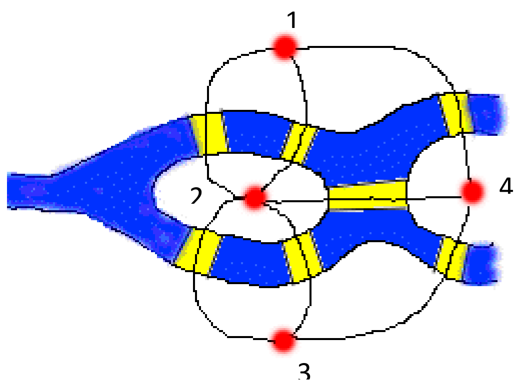
Solutions for Chapter 2

1. Eratosthenes' Problem

Let *d* be the distance between the two locations (Alexandria and Syene) and z = 7.2° is the difference in the elevation of the Sun in these cities. Now, the proportion that the arc d to the circumference of the Earth L is equal to the proportion of the angle z to the angle 360°, that is $d/L = z°/360°$, whence $L = d \times 360°/z° \approx 250{,}000$ stadia ≈ 40,000 km. Taking into account that $L = 2\pi R$, where R is the radius of the Earth, we can get $R \approx 6400$ km, which is very close to the modern value of the radius of the Earth.

2. Euler's Problem

This problem is equivalent to trying to draw the figure below without lifting your pen from the paper. The vertices 1 and 2 represent the banks of the river, the vertices 3 and 4 represent the islands and the lines represent the bridges.



We will use the following general approach. A vertex is called even if it has an even number of lines coming to (or leaving from) it. A vertex is called odd if it has an odd number of lines coming to (or leaving from) it. You can go through (i.e. come to and leave) any even vertex. If the vertex is odd, you can only start or finish at the vertex. Therefore if a figure has more than two odd vertices, it is impossible to draw it without lifting your pen from the paper. The figure above has 4 odd vertices. That is why it is impossible to draw it without lifting your pen from the paper. It means that the answer to Euler's question is 'No'.

Canterbury University in Christchurch, has incorporated a model of the bridges into a grass area between the old Physical Sciences Library and the Erskine Building, housing the Departments of Mathematics, Statistics and Computer Science. The rivers are replaced with short bushes and the central island sports a stone.

3. Reflection in a Mirror

a. Wrong idea. Look at the mirror with one eye closed. Does it make any difference? No!

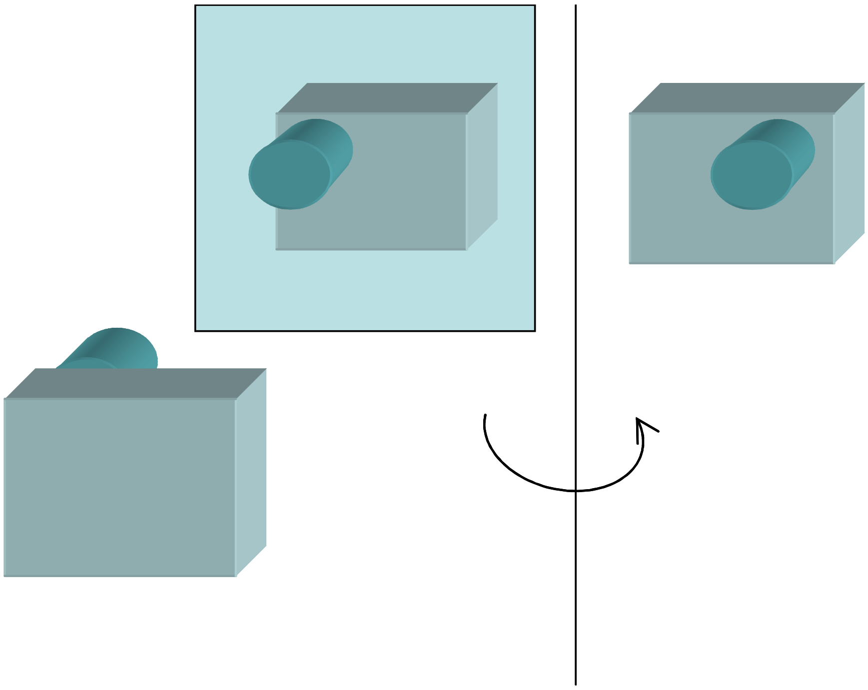
b. Wrong also. I lie in bed and look in the mirror. The guy in the mirror is lying down. His right is my left, but the top of his head is the top of my head.

c. It has nothing to do with it. You can't see your psyche in the mirror. The mirror just reflects light back. On the other hand there does seem to be an element of psychology in it (we will see it in the answer below).

d. So, clearly, none of the ideas is right. Then what is the right answer to the main question: “If mirror reverses left and right, why doesn't it also reverse top and bottom?”

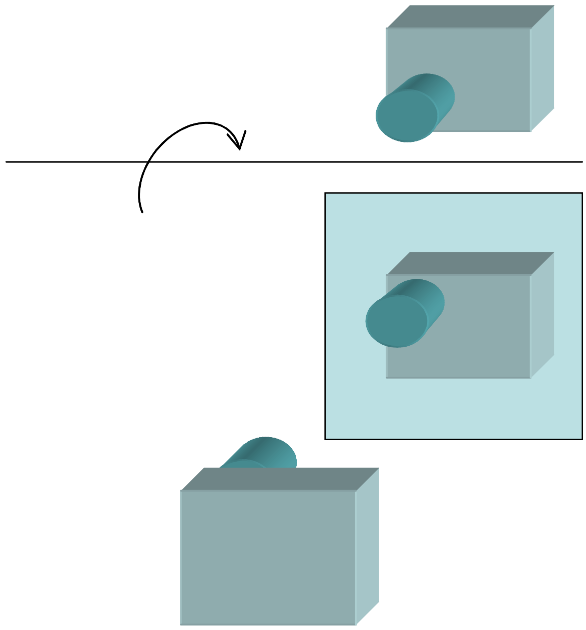
First of all, how do we know that our reflection in the mirror raises the left hand? To come to this conclusion, we imagine us turning around the vertical axis. Then we compare the mental image with what we see in the mirror.

Let us conduct an experiment. Imagine a photographic camera placed in front of a mirror. Let the camera take a photo of itself. Now let us compare the camera and its image. Just put them on the desk beside one another. We will see that the mirror reversed left and right, just as we expected.



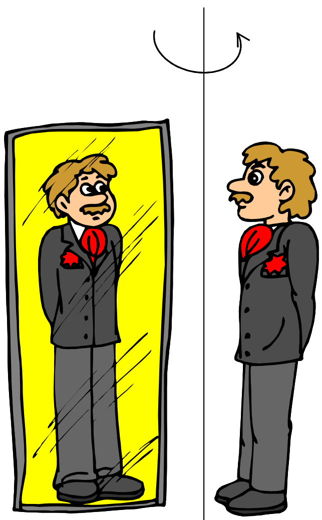
(Figure 1.) Note, in order to compare the camera with its image we turned the camera around the *vertical* axis. To compare our own image we did the same – mentally turned ourselves around the vertical axis.

Now, what will happen if we choose the horizontal axis – in other words, what if we turn the camera around the horizontal axis? The result is presented in the next Figure.



Comparing the camera with its image, we can see that it is *top and bottom* that are reversed in this case.

Now we can see that the mirror reverses images in different ways, and the type of reversal depends on the direction of the imaginary rotation axis. Certainly, it is more natural for us to imaging us turning around the vertical axis than around the horizontal axis. That is why looking in the mirror we deal with the left-right reversal.



4. 10 Types of People

If it does, you obviously belong to the type of people who understand binary code, since 10 in binary system is equal to 2.

5. A "stupid" question about bytes and bits

One bit is the smallest unit of information a computer can use – it can be either 0 or 1, corresponding to "no signal" or "signal" in computer circuitry. Characters we are used to seeing on the computer screen, such as "A", "3", "&", as well as the characters for 'space' and 'end-of-line' are stored in one byte each. Each byte consists of 8 bits. Letter "A", for example, must be written down as "01000001" in order for the computer to understand it. The 8 bit storage space allows for $2^8 = 256$ different combinations of zeros and ones. Seven bits would not be able to provide enough combinations (only $2^7 = 128$) to cover all letters, digits and symbols we use. Nine, however, would be unnecessarily too many ($2^9 = 512$).

6. "Ctrl+Alt+Del"

"Ctrl+Alt+Del" key combination was invented by an engineer from IBM, David Bradley, as a key combination which would cause a hardware interrupt to supercede serious software hang-ups. Originally designed only for support personnel, this magic trick soon became common knowledge and is now used by everyone in the world.

It is clear that to prevent computer re-boot by accident, at least three keys should be pressed simultaneously to perform this function. The reason for the choice of "Ctrl" "Alt" and "Del" is simple – these three keys are present on all computer keyboards, regardless of language or encoding (Latin, Cyrillic, Arabic, and so on…).

7. Keyboard Puzzles

This was just a joke. However, believe it or not, some people ask where the "any" key is on their keyboards when the "Press Any Key" message is displayed.

8. How many Fridays are there in February? (a puzzle from the book of Perel'man)

The usual answer is that the greatest number is five—the least, four. Without question, it is true that if in a leap year February 1 falls on a Friday, the 29th will also be Friday, giving five Fridays altogether. However, it is possible to double the number of Fridays in the month of February alone. Imagine a ship plying between Siberia and Alaska and leaving the Asian shore regularly every Friday. How many Fridays will its skipper count in a leap-year February of which the 1st is a Friday? Since he crosses the date line from west to east and does so on a Friday, he will reckon two Fridays every week thus adding up to 10 Fridays in all. On the contrary, the skipper of a ship leaving Alaska every Thursday and heading for Siberia will "lose" Friday in his day reckoning, with the result that he won't have a single Friday in the whole month. So the correct answer is that the greatest number of possible Fridays in February is 10, and the least is nil.

9. Add a Metre

Now one metre is not much of a distance, but, bearing in mind the enormous length of the Earth's orbit, one might think that the addition of this insignificant distance would noticeably increase the orbital length and hence the duration of the year. However, the result, after totting up, is so infinitesimal that we are

inclined to doubt our calculations. But there is no need to be surprised; the difference is really very small. The difference in the length of two concentric circumferences depends not on the value of their radii, but on the difference between them. For two circumferences described on a floor the result would be exactly the same as for two cosmic circumferences, provided the difference in radii was one metre in both cases. A calculation will show this to be so. If the radius of the Earth's orbit (accepted as a circle) is R, its length will be $2\pi R$. If we make the radius 1 m longer, the length of the new orbit will be

$$2\pi(R + 1) = 2\pi R + 2\pi$$

The addition to the orbit is, therefore, only 2π m = 6.28 m, and does not depend on the length of the radius. Hence the Earth's passage around the Sun, were it set 1 metre more away, would be only about 6 metres longer. The practical effect of this on the length of the year would be nil, as the Earth's orbital velocity is 30,000 m. per sec. The year would be only l/5000th of a second longer, which we, of course, would never notice.