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Project Report

Mathematical maturity: How can it inform teaching and learning of mathematics?

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Executive summary

The mathematical community considers mathematical maturity an important characteristic to have to develop as a mathematician. In this study, we investigated if mathematical maturity can be used for teaching and learning of mathematics. Following a review of the (scant) literature in this area, and a survey of professional mathematicians, we conclude that mathematical maturity can be described generically as 'expertise'. We argue that mathematical maturity is too nebulous a concept to be used meaningfully to inform teaching and learning in tertiary mathematics. An implication of this is that mathematical maturity should thus also not be used as an 'informal' prerequisite for any mathematics or statistics courses. Mathematical maturity in the sense of expertise does have its place in teaching and learning of mathematics, in particular teachers modelling, and students learning to think like a mathematician.

Introduction

Mathematical maturity is a ubiquitous concept in the mathematical community and is thought to be important to have to be able to develop as a mathematician. However, the concept of mathematical maturity is currently poorly defined. In this study, we aimed to better define the concept. Further, we sought to investigate if and how it can be taught to tertiary students. Our particular focus was on students who are not seeking to major in mathematics, but who take mathematics courses as part of their degree requirements. The central question that guided this study was: **How can mathematical maturity be defined in a measurable, and teachable way?**

To answer the question above, we did a survey of the available literature around mathematics education. The literature survey informed the development of an anonymous survey, which was sent out to professional mathematicians around the world (in particular through being promoted at the Joint Mathematics Meeting, a large international conference held in Atlanta, USA, in January 2017).

This report details the literature review, our methodology, and discusses the results of the survey and implications for the teaching of mathematics. The study was carried out within Aotearoa / New Zealand with its unique, bicultural heritage influencing the tertiary teaching and learning environment. However, the approach we took was more generic and not limited to any particular cultural or socio-political tertiary environment. Consequently, the study informs tertiary mathematics education in general, not just mathematics education within the Aotearoa / New Zealand tertiary context.

Literature review

Survey questions were developed concurrently with a cross-disciplinary review of the literature using “math maturity” or “mathematical maturity” in the title or as a key word in the abstract. Peer-reviewed journal articles and books were the primary focus of the review. However, these sources yielded few, if any clear definitions of mathematical maturity, with most authors reviewed acknowledging that mathematical maturity is poorly defined, and several arguing that mathematical maturity cannot be defined. To extend the search, blogs and websites developed by mathematicians or pertaining to mathematics pedagogy were also examined.

Despite the lack of explicit discussion about mathematical maturity, there is evidence in the literature that it is a topic with which many grapple, some in great depth and detail. In early work, Steen (1986) argued that mathematical maturity is impossible to define, arguing instead it can be conceptualised as involving two components: abstraction and synthesis.

One of the more comprehensive discussions on mathematical maturity can be found in Krantz’s (2012) book. Krantz argues that “mathematical maturity is indicative of an ability to see the big picture, to use abstraction to group ideas together in useful ways, to be able to pass back and forth between unifying concepts and specific instances. A big part of mathematical maturity is learning to pass ideas from your conscious mind to your unconscious mind” (p. 79). We interpret that as mathematical maturity not being something one achieves at a distinctive point in time, or after having taken a certain amount of coursework. Rather, the mathematically mature student is able to go beyond solving problems, and begin to create them. She or he sees *beyond* what is to new possibilities and concepts, and is able to “internalise” the building blocks. Krantz argues that mathematical maturation requires give and take interaction with peers, with models, and with masters. He does not expect undergraduate students to become mathematically mature. In fact, he doesn’t even expect most PhD students to achieve this. Mathematical maturity appears to be more for accomplished mathematicians. Other authors though, discuss mathematical maturity in high school or university students. All authors provide some list of properties that a student would exhibit to be considered mathematically mature. In Table 1 below, we summarise these properties.

Table 1: Properties of mathematical maturity

Property of mathematical maturity / becoming mathematically mature	Authors
Learn to learn mathematics, transition from rote learning to learning for understanding	Krantz (2012), Moursund (2017)
Fluency in the language of mathematics (speaking, listening, reading, writing, thinking). Looking at the language 'properties' of a problem (dimensions, expression, simplification) for clues	Moursund (2017), Stanley et al. (2004)
Approach mathematical problems from multiple perspectives (equations, function, geometric representations	Krantz (2012), Moursund (2017), Tawfeeq (2003), Hare & Phillippy (2004), Stanley et al. (2004)
Recognise and transfer mathematics in various contexts	Tawfeeq (2003), Krantz (2012), Moursund (2017), Stanley et al. (2004)
Generalise from specific examples	Krantz (2012), Moursund (2017), Stanley et al. (2004)
Communicate mathematics orally and in writing	Krantz (2012), Moursund (2017)
Recognise mathematical patterns and create your own patterns	Krantz (2012), Moursund (2017)
Represent / translate (word) problems as mathematics, recognise the mathematics in a situation	Krantz (2012), Moursund (2017), Stanley et al. (2004)
Handle abstract ideas without the need for concrete examples	Krantz (2012), Moursund (2017)
Translate concrete examples into abstract ideas	Krantz (2012), Moursund (2017), Stanley et al. (2004)
Formulate mathematical questions	Moursund (2017)
Develop intuition by abandoning naïve assumptions, separate ideas from facts, analyse hypotheses	Krantz (2012), Moursund (2017)
Recognize and analyse proofs, create your own proofs	Krantz (2012), Moursund (2017)
Develop new tools / manipulate methods to analyse new problems	Moursund (2017), Tawfeeq (2003)
Negotiate the possible parameters for a solution to a particular problem based on known axioms, theorems, and conjectures, etc.	Tawfeeq (2003)
Develop new mathematical understanding, doing independent research	Tawfeeq (2003), Hare & Phillippy (2004)
Being mathematically 'curious' and willing to experiment	Stanley et al. (2004)

Overall, terms that come up time and again on blogs and discussion forums:

- Abstraction
- Proofs
- Time, it takes time to work through a text – to read, ponder, practice, read again, ponder, practice, etc…
- Learning the language of mathematics

While Table 1 may seem like a rather lengthy laundry list, it is worth noting that there are many commonalities here with 'regular' expertise, and the properties / characteristics of experts (e.g. Bransford, Brown, & Cocking, 2000): The ability to recognise patterns, transfer knowledge from one domain to the next, seeing connections and comfort / ease in manipulating the content.

Teaching and learning for mathematical maturity

The teaching of mathematics is an area that receives a great deal of (research) attention. The bulk of this research is in the compulsory sector, with extensions to (largely US based) introductory mathematics courses (usually compulsory in the general education requirements of an undergraduate degree). As noted earlier, there is limited literature on teaching for mathematical maturity, or whether this is even a meaningful question to ask.

In his blog, Terrence Tao (2017), the 2006 Field's Medal winner and one of the world's leading mathematicians, does not explicitly mention mathematical maturity, but suggests mathematical education may be conceptualised as a series of stages that involve a move from an emphasis on computation rather than theory (stage 1), to a more formal, abstract, and theoretical one (stage 2), to one on applications, intuition, and the "big picture" frame development (stage 3). The pre-rigorous, rigorous and post-rigorous stages as he defines them typically follow, through are not limited to, the early undergraduate, to late undergraduate/early graduate, to the late graduate years and beyond. The similarities with two of the cognitive development stages as formulated by Piaget (concrete operational and formal operational) are echoed here.

The notion of stages and temporal development is one which was discussed by several authors. Krantz (2012), for example, raises the question: Is mathematical maturity something one achieves at a distinct point in time, or having passed through a select number of courses, or is it a lifelong endeavour that one never fully "achieves"?

Moursund (2017) discusses how to educate to increase mathematical maturity, and notes that there are no examples in the literature of instruments to assess mathematical maturity, and only limited teaching materials to explicitly help integrate mathematical maturity into mathematics teaching. Moursund asks how mathematics can be made relatable to students and how mathematics learning can be seen as learning a language.

The question seems to boil down to whether mathematical maturity is something that is course/content dependent/experience or skill-based, or more of a Habit of Mind (e.g. Costa & Kallick, 2008). Authors aligning with the former provide descriptors of mathematical maturity and related constructs that involved skill sets, knowledge, content and experiences. Examples included learning and being conversant in the language of mathematics, experience with and success in working with proofs, the ability to transfer knowledge within and across the discipline of mathematics into other realms, ability to move from the concrete to the abstract; to recognise and to create patterns. The latter involved approaches to problems, attitudes, or characteristics of the learner such as curiosity, tenacity, inquisitiveness, independence, self-discipline and delayed gratification.

Methodology

Based on the literature review, a survey was designed to probe professional mathematicians' ideas about mathematical maturity and if and how it could be taught. The survey was sent out via the professional connections of the authors, and was promoted at the Joint Mathematics Meeting, one of the largest professional mathematics conferences in the world, in January 2017. The questions were open-ended and were phrased as follows:

1. How is mathematical maturity defined?
2. How does one measure mathematical maturity?
3. How does one recognise mathematical maturity?
4. How does one teach for mathematical maturity?
5. How do you know if you were successful in teaching for mathematical maturity?
6. Please give an example question that you would expect a mathematically mature student to be able to answer.

In total, we received 93 surveys that contained useable data for the study. Data were analysed using a qualitative approach, chosen for its emphasis on exploring the data in depth and in great detail (Patton, 2002), so that people's perceptions, understandings, and meanings in relation to mathematical maturity could be done justice. Responses to each of the six questions were transcribed in a separate sheet in an Excel file. Participants' responses were listed in an excel sheet by row. Code development was an iterative process as data were read and analysed. During the first stage in the analysis, data were read through to gain familiarity. Upon a second read through, potential codes were identified and notes were made as data were read.

With the participant's answer to a question listed in each row, potential codes were listed in columns. Data were read through again and a number one was listed in each column if the code resonated with the participant's response. This was an inductive process such that as data were read and re-read, codes were added to or edited. The next step involved a continued iterative process of reading the data and modifying the codes. Some codes were split and some were merged during this process. Last, general themes in the responses were identified. Coding and data analysis were done by the first author for questions 1-5, and by the second author for question 6. We did not have the opportunity to do independent coding of the data by a second coder.

Results

Q1 - How is mathematical maturity defined? – 92 Responses

The vast majority of responses referred to *cognitive capabilities* as central to the notion of mathematical maturity. Referred to by some participants as "Habits of Mind", the examples provided related to cognitive capabilities ranging from thinking processes, to ability to traverse conceptual and disciplinary boundaries, to work with novel material, and a recognition of the value of mathematics in general, and mathematical principles specifically. Someone capable of abstract and/or analytic thinking was frequently described as being mathematically mature. Another commonly cited example was fluency in the language of mathematics as a defining characteristic of mathematical maturity. An understanding of how problems sit in mathematical and broader contexts was often cited in the responses to this question.

Secondary to cognitive characteristics, were responses that referred to *outcome measures* as indicative of maturity. The ability to write, solve (prove), or work with proofs was a common example cited to illustrate. Exposure to mathematical concepts and experience in mathematics were also commonly provided examples.

Students' *approach* to mathematics and mathematical problems was commonly cited as defining mathematical maturity. Examples pertaining to this theme include students' ability to apply their knowledge to existing or novel contexts, or efficiently working through problems. In addition, appropriately and insightfully asked questions were indicative of mathematical maturity, according to some responses.

Closely related in number to students' approach, were responses that discussed *characteristics of the student* as indicators of mathematical maturity. For example, mathematically mature students were considered those who display confidence, independence, rigor, patience and curiosity. One response indicated that mathematical maturity was a course dependent concept, and three responses articulated an inability or unwillingness to define the concept.

Q2 – How does one measure mathematical maturity? – 84 Responses

Tests and exam questions was the most common response to the question of how one measures mathematical maturity. Several respondents specified that questions that provided novel material to manipulate or apply were useful to gauge mathematical maturity. A much smaller number of respondents referred to specific types of questions such as those that provide counter examples, identify "flaws", justify responses, require the tester to transfer knowledge, or questions that require "logical development."

Interestingly, the same number of responses suggested either that mathematical maturity was not something that could be measured or that the respondent was unable or unwilling to answer the question about tests for mathematical maturity. Two participants felt that mathematical maturity was something one knows when it is seen. Only one response articulated that the distribution of marks/grades will indicate mathematical maturity among students.

Face-to-face discussions and/or task-based interviews were referred to by several respondents as a useful way to measure mathematical maturity. One response suggested a similar vein - that mathematical maturity is best tested as one would test for language fluency.

Engagement in the (United States) Common Core Curriculum practise standards was mentioned in one response as a means to measure mathematical maturity.

Q3 – How does one recognise mathematical maturity? – 75 Responses

The most common response to the question of how one recognises mathematical maturity referred to personal communication, one-on-one interactions with, and observations of students. Some respondents specifically referred to the observation of students engaging in in-class assignments. Other respondents articulated that they will know [mathematical maturity] when they see it." Others referred to the value of employing a "holistic' assessment of mathematical maturity rather than a step-by-step progression towards mathematical maturity.

The remaining responses were somewhat evenly spread across the following four categories: Product or Outcome measures; Student cognitive processes; Students approach to tasks; and Characteristics of the student.

Responses related to *student cognitive processes* included evidence that students could 'handle' or apply novel material. In addition, students' ability to demonstrate transfer of concepts within and across the discipline was mentioned. Understanding basic concepts was referred to in four responses for this question. An equal number of responses indicated students with mathematical maturity recognise the value of solutions. A more specific set of skills, such as success with writing, tackling proofs was mentioned.

Approach-related responses included evidence that students asked the "right" questions, and could utilise multiple strategies.

Product/outcome related responses included examples such as the ways in which students demonstrate that previous questions have been absorbed and/or applied, students' ability to solve complex problems, the facility with which students handle or solve problems. In addition, some responses referred to in-class assignments as providing a stage on which mathematical maturity can be recognised. Writing was a product/outcome measure indicated in more than a few responses (i.e. 'coherency of written work' or the ability to produce mathematical texts) to how one identifies mathematical maturity.
Five respondents argued that they were either unable or unwilling to recognise or define mathematical maturity.

Confidence and independence were two *characteristics* of a mathematically mature student.

Q4 – How does one teach for mathematical maturity? – 71 Responses

The question of how to teach for mathematical maturity yielded a wide range of responses. The largest number of responses referred to modelling or "showing how a mathematician thinks". Second in popularity were responses that referred to the "big" picture as well as underlying principles behind mathematics in general and tasks they were facing in particular; in other words, to teach for meaning and purpose. Related to this were responses that referred to "working through problems" in lectures or demonstrating either the application of principles or techniques.

Some responses referred to specific tasks such as engaging students with application problems of increasing complexity, to work collaboratively, to engage with novel material, or by asking students to work through and/or write proofs and induction problems, to have students write - critical reviews or otherwise. Other responses referred to Habits of Mind or value sets to develop such as encouraging in students a "desire for practice". Time and experience, and "internalisation and actualisation of knowledge" were also mentioned by participants as necessary components of an education for mathematical maturity - these being more prominent components than specific tasks or skill sets.

Interestingly, logic was one aspect of mathematics mentioned in three responses as something one should "teach" for mathematical maturity.

Q5 – How do you know if you were successful in teaching for mathematical maturity? – 66 Responses

Outcome measures such as student performance on exams, success in tackling problems, and course marks/grades were most commonly cited as evidence that teaching for mathematical maturity was successful. On a similar vein, some participants cited post-graduate performance, advancement, and work or school placement of graduates as an indicator of success. A close number of responses referred to communication with the student as the means by which one gauges the success of mathematical maturity-directed teaching.

The next most common answer type aligned with one of a number of responses including: "It's hard to tell the source of maturity", "I'm unsure", "It wasn't”, "I can't," and "I don't".

Several participants wrote of seeing evidence of depth of understanding and subtlety in approach and engagement with the material. A similar number identified the writing of original proofs or student performance working with proofs as an indication of mathematical maturity. Several more specific examples were cited. These include observing students ask questions that reflected mathematical maturity, students' abilities to solve non-routine problems; to make connections, to volunteer alternate approaches.

In addition, student independence and student persistence were cited as indicators. Relating to the characteristics and attitudes of the students, two respondents referred to student satisfaction/interest and engagement in work as a reflection of students gaining mathematical maturity.

Observation of students was indicated by a small number of respondents to this question. On a similar vein, two respondents noted, "I know when I see it."

Q6 – Example Question – 68 Responses

The most common type of question makes heavy use of informal proof and explanatory techniques. Problem solving ability, an understanding of concepts and their relationships, and ability to "explain" properties of mathematical structures. More than half of the response questions included asking for informal proofs (proof outlines with limited symbolic reasoning) and problem solving.
About a third of responses ask for more formal proofs of theorems. Typically, these questions involve fundamental notions in the respective branch of mathematics from which they are drawn; basic properties of numbers, functions, and sets.

When specific branches of mathematics were involved, algebra and analysis were two branches that stood above the others. Example questions were relatively equally distributed among other branches. This suggests that the ability to deal with the abstract is seen as evidence for an underlying mathematical maturity. Surprisingly few responses dealt with actual application of mathematics (fewer than 10 out of the 68 responses were application-based).

What was even more surprising, given that interaction and communication was the most common response to how mathematical maturity is recognised, is that only one response question asks the student to explain mathematics to less advanced students.

In terms of what is needed to answer these questions, the vast majority (more than 50) required an understanding of mathematical proof. Most of these require an in-depth conceptual knowledge of the basic building blocks of some branch of mathematics (for example, convergence of sequences in analysis). There were 28 responses that require a deep understanding of the role of abstraction and a familiarity/comfort with manipulating symbols (ability to translate natural language into mathematical symbols, and manipulating those symbols).

Discussion and conclusions

The responses to the survey align with the literature in the sense that there is at best a broad, yet very general agreement on what mathematical maturity is. A person with mathematical maturity, according to the data in this study, is

- Conversant, if not fluent, in the language of mathematics
- Capable of abstract/conceptual/logical thinking - lateral/horizontal thinking - integrating the 'novel' with the 'known'
- Proof competent
- One who possesses certain habits of mind - diligence, stamina, inquisitiveness/curiosity, patience, and the ability to work both independently and collaboratively
- Independent and capable of working with and learning from others
- Appreciative of the 'value' of mathematics across contexts
- Able to articulate what they know forwards and backwards orally, in writing, and through worked examples
- Capable of engaging with non-routine problems and novel material

We argue that as a concept, mathematical maturity has considerable similarity with the general concept of expertise. Experts display characteristics such as fluency in the topic, being able to quickly recognise patterns, and transfer knowledge from one area to the next.

Results also showed a wide range of responses to the teaching for mathematical maturity. Again, these techniques, such as modelling expert thinking, one-on-one interaction, nurturing the student towards independence, and questioning / practice / engagement / demonstration are both general good pedagogical practices, and techniques to help students become more expert-like.

Casting mathematical maturity generically as 'expertise' also explains the variety of opinions on whether mathematical maturity is situated (i.e. dependent on the context) or an absolute, stand-alone quality that is achieved at some point in time. We would argue that indeed, mathematical maturity needs to be situated. The level of expertise in mathematics is not a static construct, but is context (e.g. age, experience) dependent.

We conclude that mathematical maturity is not a meaningful construct to use in (the design of) teaching and learning in tertiary mathematics. It is too nebulous a construct to be operationalised in a way that can inform classroom practice. From an educator's perspective, we think that a more fruitful conversation would be about what we want our students to understand and to be able to do as they progress through the curriculum in mathematics. That is, a focus on learning outcomes and constructively aligned assessments and teaching and learning activities, rather than using a vague construct like mathematical maturity. Such a discussion will have different outcomes depending on the socio-cultural context in which mathematics is taught (e.g. Aotearoa / New Zealand's bicultural heritage) as well as for example requirements set by external accrediting bodies in disciplines that rely heavily on mathematics (service) teaching (e.g. Engineering New Zealand (formerly IPENZ)).

Consequently, we strongly caution against the use of mathematical maturity as a formal or informal prerequisite for tertiary mathematics (or statistics) courses.

That said, mathematical maturity in the sense of expertise has its place in teaching. This is not just through the obvious required content knowledge to teach the subject material, but in particular through expert modelling. Teachers, in their own teaching behaviours, can be explicit in showing students how an expert mathematician thinks about and approaches mathematical problems, allowing students to learn to think like a mathematician.

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